CHARACTERIZATION OF RINGS USING QUASIPROJECTIVE MODULES

BY

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ABSTRACT

Semisimple, semiperfect, and perfect rings are characterized by quasiprojective modules and quasiprojective covers over them.

In this paper R will always denote an associative ring with 1 and all modules and morphisms will be taken from the category of unitary left R-modules. "Semi-simple" will mean "Jacobson semisimple and artinian.'

A module Q is called *quasiprojective* iff every diagram

$$\begin{array}{ccccc} & & Q & & \\ & & \downarrow{\scriptstyle\alpha} & & \\ Q & \xrightarrow{\beta} & M & \longrightarrow & 0 \end{array}$$

can be embedded in a commutative diagram

$$\begin{array}{ccccc} & & Q & & \\ & {\scriptstyle\gamma}\swarrow & \downarrow{\scriptstyle\alpha} & & \\ Q & \xrightarrow{\beta} & M & \longrightarrow & 0 \end{array}$$

Such modules were first studied by Miyashita [6] and by Wu and Jans [8]. Projective modules are clearly quasiprojective, as are completely reducible modules.

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An epimorphism $\alpha : P \to M$ is a *projective cover* of the module M whenever P is projective and $\ker(\alpha)$ is small in P (A is small in B iff $A + C = B$ implies $C = B$). In an analogous manner Wu and Jans defined a *quasiprojective cover* to be an epimorphism $\beta : Q \to M$ such that (i) Q is quasiprojective; (ii) $\ker(\beta)$ is small in Q; and (iii) Q/T is not quasiprojective for all $0 \neq T \subseteq \ker(\beta)$.

1. Semisimple rings. We will first characterize semisimple rings using quasi-projectives and dualizing the result of Faith and Utumi [2] for quasi-injectives.

LEMMA 1.1. *Let P be projective and Q quasiprojective. Then a sufficient condition for an exact sequence* $0 \to K \to P \xrightarrow{\lambda} Q \to 0$ *to split is that* $P \oplus Q$ *be quasiprojective.*

PROOF. Assume $P \oplus Q$ is quasiprojective and define epimorphisms α, β: $P \oplus Q \to Q$ by $\alpha(p,q)\alpha = q$, $(p,q)\beta = p\lambda$. By quasiprojectivity there then exists an endomorphism θ of $P \oplus Q$ making the diagram

$$\begin{array}{ccccc} & & P \oplus Q & & \\ & \theta \swarrow & \downarrow \alpha & & \\ P \oplus Q & \xrightarrow{\beta} & Q & \longrightarrow & 0 \end{array}$$

commute. Define $\gamma : Q \to P$ by $q\gamma = (0,q)\theta\pi$, where $\pi : P \oplus Q \to P$ is the canonical projection. Then for all $q \in Q$, $q\gamma\lambda = (0,q)\theta\pi\lambda = (0,q)\theta\beta = (0,q)\alpha = q$. Thus $\gamma\lambda$ is the identity on Q and so the sequence splits.

COROLLARY 1.2. *A sufficient condition for R to be semisimple is that $R \oplus M$ be quasiprojective for every simple module M.*

PROOF. If M is simple then there exists an exact sequence $0 \to K \to R \to M \to 0$ which splits by Lemma 1.1 (simple modules being quasiprojective). Therefore every simple module is projective which implies that R is semisimple.

THEOREM 1.3. *The following are equivalent:*

(1) *R is semisimple.*

(2) *The class of quasiprojective modules is closed under finite direct sums.*

(3) *Every module is quasiprojective.*

(4) *Every finitely-generated module is quasiprojective.*

PROOF. Since (1) implies that every module is projective, (1) implies all of the other statements. By Corollary 1.2, (2) and (4) imply (1). (3) implies (4) trivially.

2. Semiperfect rings. Bass [1] defines a ring to be semiperfect if and only if every finitely-generated left module over it has a projective cover.

LEMMA 2.1. *If $\alpha: P \to M$ is a projective cover then $\bar{\alpha}: P/T \to M$ is a quasi-projective cover, where T is the (unique) maximal submodule of* ker(α) *stable under endomorphisms of P and $\bar{\alpha}$ is canonically induced by α.*

PROOF. See [8], Proposition 2.6.

THEOREM 2.2. *R is semiperfect iff every finitely-generated module has a quasiprojective cover.*

PROOF. Sufficiency follows directly from Lemma 2.1. Hence we have to show necessity. Let M be a finitely-generated module and let $0 \to K \to F \xrightarrow{\gamma} M \to 0$ be exact with F finitely-generated and free. By (2) $F \oplus M$ has a quasiprojective cover $\alpha: Q \to F \oplus M$. Denote the canonical projection $F \oplus M \to F$ by π. Since F is projective the exact sequence $0 \to L \to Q \xrightarrow{\alpha\pi} F \to 0$ splits ($L = \ker(\alpha\pi)$) and so there exists $\lambda: F \to Q$ such that $\lambda\alpha\pi = \mathrm{id}_F$, $Q = F\lambda \oplus L$. Without loss of generality we can therefore take $Q = F \oplus L$.

Let α' be the restriction of α to L; then α' is an epimorphism onto M. We claim that in fact $\alpha': L \to M$ is a projective cover. Certainly $\ker(\alpha')$ is small in L since, if we suppose $\ker(\alpha') + A = L$, then $A = F \oplus L = [F \oplus A] + \ker(\alpha)$ so $F \oplus A = Q = F \oplus L$, implying $A = L$.

By the projectivity of F, there exists a homomorphism $\beta: F \to L$ making the diagram

$$\begin{array}{ccccc} & & F & & \\ & \swarrow^{\beta} & \downarrow \gamma & & \\ L & \xrightarrow{\alpha'} & M & \longrightarrow & 0 \end{array}$$

commute. Since γ is an epimorphism, $L = \mathrm{im}(\beta) + \ker(\alpha') = \mathrm{im}(\beta)$ by the smallness of $\ker(\alpha')$ and so β is an epimorphism. By Lemma 1.1 the sequence $0 \to \ker(\beta) \to F \to L \to 0$ splits, proving L isomorphic to a direct summand of F and hence projective.

Note that in the proof above we made no use of the fact that M was finitely-generated other than to obtain a quasiprojective cover and that we did not make use of property (iii) of the definition of quasiprojective cover.

3. Perfect rings. A ring R is *left perfect* if and only if every left R-module has a projective cover. Bass ([1], Theorem P) proves several conditions equivalent to left perfectness, among them: (a) R satisfies the descending chain condition on principal right ideals; (b) Every flat module is projective.

THEOREM 3.1. *The following are equivalent;*

(1) *R is left perfect.*

(2) *Every module has a quasiprojective cover.*

(3) *Every flat module is quasiprojective.*

PROOF. (1) $\Rightarrow$ (2) follows from Lemma 2.1 and the proof of (2) $\Rightarrow$ (1) is identical with that of (2) $\Rightarrow$ (1) of Theorem 2.2. (1) $\Rightarrow$ (3) follows, as noted, from Bass' Theorem P. We are thus left to show (3) $\Rightarrow$ (1).

By Theorem P we have to show that R satisfies the descending chain condition on principal right ideals or equivalently that for every sequence $\langle a_i \rangle$ of elements of R there exists an m such that $a_1 \cdot \cdots \cdot a_m R = a_1 \cdot \cdots \cdot a_{m+k} R$ for all $k \geqq 0$. Let $F = \oplus_{i=1}^{\infty} Rx_i$ be a countably-generated free module and define $G_n = \oplus_{i=1}^{n} R(x_i - a_i x_{i+1})$ $(n = 1, 2, \cdots, \infty)$ Then F/G_n is free (hence flat) for all $n < \infty$ and so $F/G_\infty = \lim_{\rightarrow} F/G_n$ is the direct limit of flat modules and so is flat. F itself is flat and hence so is $F \oplus F/G_\infty$. By (3) $F \oplus F/G_\infty$ is quasiprojective and so by Lemma 1.1, G_∞ is a direct summand of F. This suffices to prove what we need by Lemma 1.3 of [1].

In closing we should note a result due to Sandomierski [7]: R is left perfect [semiperfect] iff every completely reducible [simple] module has a projective cover. Since simple and completely reducible modules are quasiprojective, this shows in particular that R is left perfect [semiperfect] iff every quasiprojective [finitely-generated quasiprojective] module has a projective cover.

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